Formula for 1D Perfectly Elastic Collisions

Assume that masses m_1 and m_2 are moving along the x-axis at velocities v_1 and v_2, respectively. Then they collide. Conservation of momentum tells us:

$$m_1 v_1 + m_2 v_2 = m_1 \tilde{v}_1 + m_2 \tilde{v}_2$$

where v indicates a velocity before the collision, and $\tilde{v}$ indicates a velocity after the collision.

Since we are assuming a perfectly elastic collision, it must also be the case that the total kinetic energies before and after the collision are equal. We have:

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 \tilde{v}_1^2 + \tfrac{1}{2} m_2 \tilde{v}_2^2$$

We thus have two equations in two unknowns, and the rest is merely algebra. However, let's write out the "merely", since it is a bit involved. To save keystrokes, I will define $\mu = m_1/m_2$, and then I will divide through both equations by m_2. This yields:

momentum equation: $\mu v_1 + v_2 = \mu \tilde{v}_1 + \tilde{v}_2$

kinetic energy equation: $\mu v_1^2 + v_2^2 = \mu \tilde{v}_1^2 + \tilde{v}_2^2$

Regrouping both equations:

1) $\mu(v_1 - \tilde{v}_1) = \tilde{v}_2 - v_2$

2) $\mu(v_1^2 - \tilde{v}_1^2) = \tilde{v}_2^2 - v_2^2$

We now note that $x^2 - y^2 = (x + y)(x - y)$, so equation (2) becomes:

3) $\mu(v_1 + \tilde{v}_1)(v_1 - \tilde{v}_1) = (\tilde{v}_2 + v_2)(\tilde{v}_2 - v_2)$

We can divide both sides of equation (3) by the respective side of equation (1) to get:

4) $v_1 + \tilde{v}_1 = \tilde{v}_2 + v_2$, which is a pretty amazing equation, since it tells us that we have a sort-of conservation of velocity between the two masses.

We then solve equation (4) for $\tilde{v}_2$, and substitute back into equation (1):

5) $\mu(v_1 - \tilde{v}_1) = (v_1 + \tilde{v}_1 - v_2) - v_2$, or $\mu v_1 - \mu \tilde{v}_1 = v_1 + \tilde{v}_1 - 2v_2$

A little rearrangement yields $\tilde{v}_1 = [2v_2 + (\mu - 1)v_1] / (1 + \mu)$. Finally, multiplying top and bottom by m_2 gives us:

$$\tilde{v}_1 = [2m_2 v_2 + (m_1 - m_2)v_1] / (m_1 + m_2)$$

Since it is evident that the masses must be symmetric under mirror inversion, we can immediately write down an expression for the velocity of the second mass*:

$$\tilde{v}_2 = [2m_1 v_1 + (m_2 - m_1)v_2] / (m_1 + m_2)$$

*This is the pompous sort of thing that physicists say in their publications *after* they've worked through all the algebra and realize in retrospect that the solution for the second velocity is the same as the solution for the first velocity, except with all the subscripts (1,2) reversed.