

In class on Friday (October 30) I pointed out that rotational *motion* can be treated exactly like linear motion. The only difference is that you use $\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha_T t^2$ rather than $x = x_0 + v_0 t + \frac{1}{2} a t^2$. As an example, I took a disk freely rotating at ω_0 . If you decide to stop the disk by just placing your hand against it, then you are exerting a frictional force of $F = \mu N$, where N is how hard you are pressing down. The friction will oppose the motion, giving an acceleration of $-\alpha_T$. The time it will take the disk to come to a halt is $\omega_0 = \alpha_T t$ (like $v = at$), and the number of radians that the disk will turn before it stops is $\theta = \frac{1}{2} \alpha_T t^2 = \frac{1}{2} \alpha_T (\omega_0 / \alpha_T)^2 = \frac{1}{2} \omega_0^2 / \alpha_T$, like $x = \frac{1}{2} a t^2 = \frac{1}{2} v^2/a$.

However, the *dynamics* of linear and rotational motion are not so easily reconciled. Newton's Laws are still enforced in rotational motion, but the details become murky. For example, suppose we wanted to relate the α_T in the above equations to the frictional force, μN . Can we just use $F = ma$ as our model and say that $F = \mu N = M \alpha_T$, where M is the mass of the disk?

Hmn, no. For one thing, force has the dimensions of kg m/s^2 , and $M \alpha_T$ has the dimensions of kg radians/s^2 . Not the same thing.

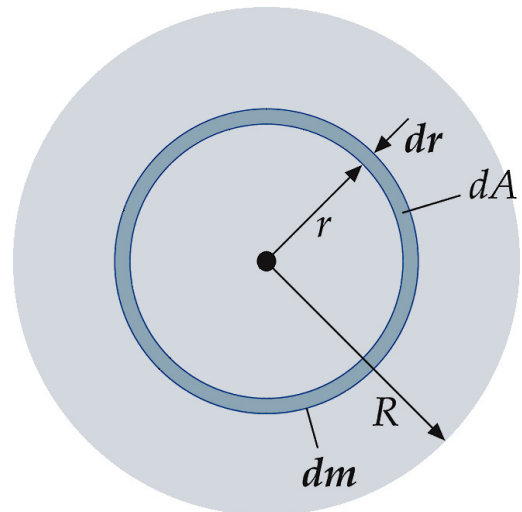
Right. Well, we know that we can convert radians to meters using $x = \theta R$, and if we take a couple of time derivatives we have $a = \alpha_T R$. How about using $F = \mu N = ma = M \alpha_T R$? At least the dimensions are OK.

Hmn, no. The problem is, if you throw a disk in a straight line with no rotation, then every atom in the disk is moving at the same speed. But if you have a rotating disk, then points on the outer edge are moving faster than points farther in. All the atoms in a solid rotating disk have the same *angular* speed, but angles do not create kinetic energy. Meters per second create kinetic energy.

When you have a group of particles that aren't moving at the same speed, the only way to calculate their total kinetic energy is to add up the energy from each particle. In the case we have here, which is a solid uniform disk of mass M and radius R , we will need to set up an integral.

Let us consider an infinitesimal mass dm , located anywhere in the disk that you like. It is carrying an infinitesimal kinetic energy of $dE = \frac{1}{2} dm v^2 = \frac{1}{2} dm \omega^2 r^2$, where r is the (variable) distance out to the mass. A bit of reflection tells us that we can extend dm into a ring all the way around the disk, as shown below, because all the points on this ring are at the same radius and therefore have the same velocity ($v = \omega r$) and therefore the same kinetic energy. Now, the radius r is obviously our variable in this problem, so we need to convert dm into some function of r to do the integral. We can do this by taking a ratio. Since the disk is uniform, the ratio of dm to the total mass must be the same as the ratio of dm 's area to the area of the entire disk. In symbols: $dm/M = (2\pi r dr) / \pi R^2$. The area occupied by dm is equal to $2\pi r$ (like a highway going around the disk), multiplied by an infinitesimal width, dr .

[One can also derive this differential area by noting that $A = \pi r^2$ for a disk, so $dA = 2\pi r dr$ is the area of an infinitesimal strip added to the edge of the disk.]



Inserting dm back into the equation for kinetic energy gives us $dE = \frac{1}{2}[M(2\pi r dr)/\pi R^2]\omega^2 r^2 = (M/R^2)\omega^2 r^3 dr$. We now integrate from 0 to R to get: $E = \frac{1}{4} (M/R^2)\omega^2 r^4 \Big|_0^R = \frac{1}{4} MR^2\omega^2$.

The work done by your hand as you hold it on the edge of the spinning disk is not subject to the problem of multiple speeds. You are only touching the outer rim, so the distance that your hand moves along the disk is just $x = \theta R$. Therefore, the frictional work done is $W = Fd = \mu N \theta R$. The disk will come to a stop when this work equals the total initial kinetic energy of the disk, which we have just calculated, so $\mu N \theta R = \frac{1}{4} MR^2\omega^2$.

We now take a time derivative of both sides, and remembering that $\omega = d\theta/dt$ and $\alpha = d\omega/dt$, we have $\mu N \omega R = \frac{1}{4} MR^2 (2\omega\alpha)$, or $\mu N = \frac{1}{2} M\alpha R$. So, after all that, we see that the frictional force and the acceleration are not related by $\mu N = M\alpha R$, as one might guess just by robotically plugging symbols into $F = ma$, but rather, by $\mu N = \frac{1}{2} M\alpha R$.

In this case. Lest you mislead yourself into believing that you could have guessed that factor of $\frac{1}{2}$ by averaging $r = 0$ and $r = R$ to get $r = \frac{1}{2}R$ (or something like that), allow me to assure you, this hideous calculational machinery was necessary to get that factor of $\frac{1}{2}$. The machinery generates a factor of $\frac{1}{2}$ for a uniform disk, but it will generate something entirely different for a rotating square, a rotating beam, a rotating sphere, and so on.

We will study a better way of dealing with rotational energy when we look at the concept of moment of inertia.