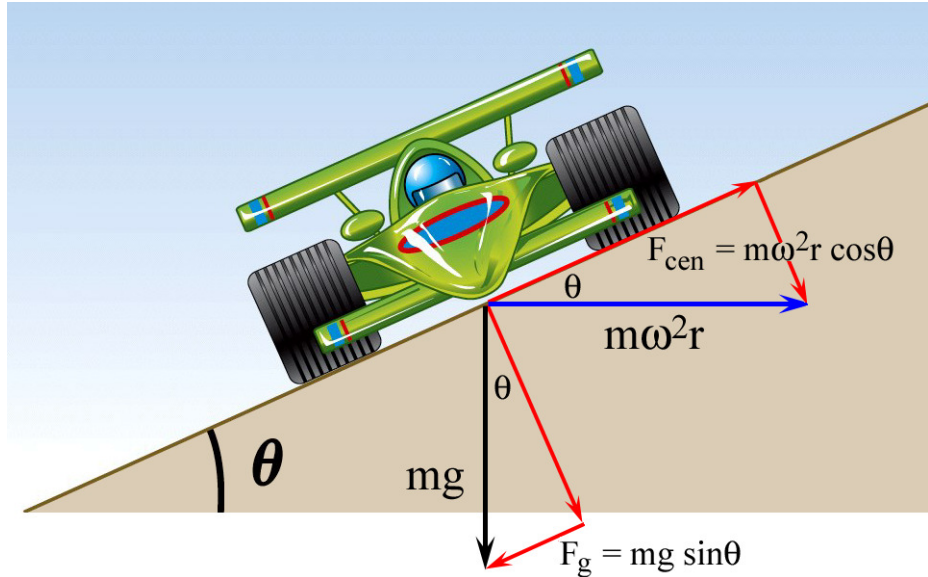


Race Car on a Track

Let's consider the motion of a race car as it goes around a circular, banked track. The question is, what's the top speed the car can achieve without sliding up and off the top of the embankment? The physical situation is shown in the illustration.



We have three forces to deal with: gravity, friction, and the so-called centrifugal force. (See below for rant.)

Using the relation that

$$v = (2\pi r)f = r\omega$$

for circular motion, where ω is the angular frequency, we can say that $F = m\omega^2 r$ for the centrifugal force, and this is the force vector shown in blue in the illustration.

Calculating the top speed is a matter of balancing vectors. The gravity pulling the car down the slope and the friction acting on the tires must equal the “centrifugal force” acting to push the car up the slope. (**Rant:** Talk about your awful terminology. There is no fictional “centrifugal force” pushing on the car. All we have is a situation where the forces acting to accelerate the car in a circle – gravity plus friction – have limits on how strong they can be. If the driver pushes the car’s speed to the point where the acceleration needed to move it in a circle exceeds what gravity plus friction can supply, then the car will stop moving in a circle and skid more-or-less in a straight line, which means it will leap off the top of the embankment in a direction tangential to the embankment. The car is not being “pushed” upward or outward by anything. But those are the words we use, because the magnitude of the acceleration of anything moving in a circle is v^2/r , and if you multiply by m then you have “centrifugal force” = mv^2/r . Bah, humbug.)

The gravity mg acting on the car pulls downward along the y-axis as shown. Only the vector component of the gravity acting down the slope can accelerate the car, and the usual trigonometry gives us $F_G = mg \sin\theta$ for this component. An object moving in a circle must have an acceleration vector that points directly towards the center of the circle, or to put it another way (and as much as I loath putting it this way), the centrifugal force on the car must point directly outward from the rotation axis. In this case the car is circling the center of the race track, which is off to your left in the illustration, so the centrifugal force points directly away from that, along the positive x-axis as shown. Therefore the component of the centrifugal force that is acting along the ramp is $F_{cen} = m\omega^2 r \cos\theta$.

As for the friction, we have $F_f = \mu N$. The normal force N is the force acting directly into the slope, and in this case it is being generated by two sources: gravity and the centrifugal force. Contemplation of the illustration and some basic trig tells us that the normal component due to gravity is $mg \cos\theta$, and the normal component due to centrifugal force is $m\omega^2 r \sin\theta$.

The car will begin to slide off the track when the centrifugal force acting along the track just equals gravity and the frictional force. We have:

$$\begin{array}{ccccccc}
 m\omega^2 r \cos\theta & = & mg \sin\theta & + & \mu [& mg \cos\theta & + & m\omega^2 r \sin\theta] \\
 \text{(sloped component} & & \text{(sloped component} & & \text{(normal component} & & \text{(normal component} \\
 \text{of centrifugal force)} & & \text{of gravity)} & & \text{of gravity} & & \text{of centrifugal force)}
 \end{array}$$

As it often does in problems that involve a single mass plus friction, gravity, or centrifugal force, the mass cancels out. The size of the car doesn't matter, only how good its tires are. A bit of algebra gives us the final answer: $v^2 = gr (\mu + \tan\theta) / (1 - \mu \tan\theta)$

We can also ask the opposite question: what is the *minimum* speed the race car must go so that it will not slide *down* the track? There are two answers to this. If the coefficient of friction is high enough, then the minimum speed is zero because the friction is greater than gravity. You will no doubt recall that we worked through a similar example in class, and found that there is a critical angle at which an object setting on a ramp will just barely begin to slide: $\tan\theta_c = \mu_s$. If θ is less than θ_c , then the race car cannot slide down the ramp. For $\theta > \theta_c$, the gravity is counteracted by both the centrifugal force and the friction, or:

$$\begin{array}{ccccccc}
 mg \sin\theta & = & m\omega^2 r \cos\theta & + & \mu [& mg \cos\theta & + & m\omega^2 r \sin\theta] \\
 \text{(sloped component} & & \text{(sloped component} & & \text{(normal component} & & \text{(normal component} \\
 \text{of gravity)} & & \text{of centrifugal force)} & & \text{of gravity} & & \text{of centrifugal force)}
 \end{array}$$

A bit of algebra yields: $v^2 = gr (\tan\theta - \mu) / (1 + \mu \tan\theta)$, which is the same as our answer for the car sliding up the ramp, except that we've changed the sign of μ .