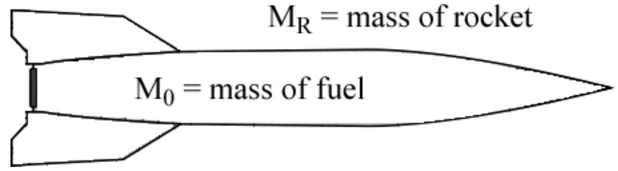
The Simple Rocket Equation II



Let's derive the rocket equation again, this time using conservation of momentum rather than Newton's Laws. We again have a rocket of mass M_R filled with a mass M_0 of fuel. The exhaust velocity of the fuel relative to the rocket is v_0.

In the previous derivation, I had to assume a constant rate of fuel burning ($= R$) to make the math manageable. Here all we need to do is note that a small amount of fuel dM which is exhausted at a speed of v_0 will carry a small amount of momentum, $p = dM\ v_0$. The rest of the rocket, whose mass M is unknown because we don't know how much fuel has been burnt, will gain a small, equal, opposite momentum, $p = M\ dv$. The initial momentum of the system is zero, so conservation of momentum gives us $0 = dM\ v_0 + M\ dv$. A bit of algebra yields $-dv = v_0\ (dM/M)$, and then a bit of calculus yields $-v = v_0 \ln(M) + C$.

We evaluate the constant of integration by noting that the mass of the rocket is $M_R + M_0$ when $v = 0$. This gives $C = -v_0 \ln(M_R + M_0)$, and with a bit of algebra, $v = v_0 \ln[(M_R + M_0)/M]$.

This result gives us the velocity of the rocket at any time as a *function* of its *current* total mass, regardless of the rate or manner in which its fuel has been expended! If we want to evaluate the formula for the final velocity, we just note that $M = M_R$ when all the fuel is gone, so

$$v_F = v_0 \ln[(M_R + M_0)/M_R] = v_0 \ln(1 + M_0/M_R).$$

This is exactly the same formula that we derived previously.