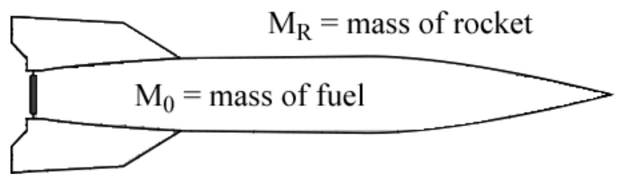
The Simple Rocket Equation

Consider a simple rocket. This is one that has a lot of fuel, a hull to hold the fuel, and not much else. (A pretty good description, by the way, for both very large rockets like the Saturn V, and very small rockets like the 3-inch ones sold as home fireworks.



Sheer speed is almost the only criteria for these guys, so they are basically the Newtonian equivalents of drag-racers: long on propellant and very short on manuverability.)

Question: if I take a simple rocket to a weightless environment and let it burn up all its fuel, how fast will it be going?

From an action-reaction standpoint, a rocket is like a cannon firing a cannonball. The cannonball goes one way, and the cannon recoils the opposite way. However, the rocket burns off its reaction mass gradually, not in one burst, so we must resort to calculus to find the final rocket velocity. We will let M_R = the mass of the rocket and M_0 = the mass of the fuel. We can assume that the fuel burns at a constant rate of $dM/dt = R$, and exits the rocket with a constant velocity of v_0. The thrust (force) exerted by the burning fuel is $F = Rv_0$.

Note – As discussed in class, the most general expression for force puts the mass *inside* the time derivative with the velocity: $\mathbf{F} = d(m\mathbf{v})/dt = m(d\mathbf{v}/dt) + (dm/dt)\mathbf{v}$, where I've used the usual chain rule of differentiation. The first term in the chain is our old friend $m\mathbf{a}$, and the second term is none other than Rv_0, if you compare it to the constants we defined for our rocket fuel. The term $(dm/dt)v$ is called *thrust* by physicists, but don't let the special word fool you – thrust is just a force. It isn't hard to list common examples: household fans generate thrust, as do propellers on ships and airplanes, jet engines, and even the water spray in a kitchen sink or shower.

Returning to the rocket problem, we simply apply $\mathbf{F} = m\mathbf{a}$ to the whole system, which in this case means thrust = [rocket mass + fuel mass]($d\mathbf{v}/dt$). I have converted $\mathbf{a}$ into $d\mathbf{v}/dt$ because we want to know the rocket's final velocity, not its acceleration.

We know the thrust = constant = Rv_0, so that leaves us with the rocket's mass. It starts off with a total mass of $M_R + M_0$, but this dwindles as the fuel is burnt. Since the rate of burning is constant, we know that $R \times t$ of the fuel will be burnt in 't' seconds. Thus, the mass of the fuel left in the rocket at any time is $M_0 - Rt$. The equation for the velocity becomes $Rv_0 = (M_R + M_0 - Rt)(d\mathbf{v}/dt)$. To get from this to the final velocity we only have to collect terms to put dv on one side of the equation and dt on the other, then integrate: $Rv_0\, dt / [M_R + M_0 - Rt] = dv$, or $(-{}^{1}/_{R})Rv_0 \ln(M_R + M_0 - Rt) = v + C$. We can evaluate the constant of integration C by noting that $v = 0$ at $t = 0$. This gives $-v_0 \ln(M_R + M_0) = C$.

The resulting equation $-v_0 \ln(M_R + M_0 - Rt) = v - v_0 \ln(M_R + M_0)$ gives us the velocity of the rocket at any time t. To find the final velocity, we note that the fuel must run out when $Rt_c = M_0$, so inserting $t_c = M_0/R$ gives us $-v_0 \ln(M_R + M_0 - R \times M_0/R) + v_0 \ln(M_R + M_0) = v$, or $v_F = v_0 \ln(1 + M_0/M_R)$.

The red equation is known as the *Simple Rocket Equation*. Two points about it are noteworthy. First, the rate R at which the fuel is burned does not appear in it! As far as the ultimate velocity of the rocket is concerned, it does not matter if the force exerted by the fuel comes in an extended blazing blast or in a series of gentle puffs spread out over a century. The total momentum is the same in either case.

Second, that logarithm is a killer, if you are a rocket scientist. If your rocket consists of 50 tons of payload and 500 tons of fuel ($M_0 = 10M_R$), then your final speed will be $\ln(11) = 2.4$ times v_0. But if you decide you need super-speed, and so you build a stupendous mega-rocket which has the same 50-ton payload, but 50,000 tons of fuel, then your final speed will be $\ln(1001) = 6.9\ v_0$.

Good grief. You burn *100* times as much fuel, and probably spend 500 times as much money building a huge rocket to carry all the fuel, and what do you get? A paltry speed increase of $2.4\ v_0$ to $6.9\ v_0$.

The problem with a rocket is, the larger you make the rocket then the more fuel you have to burn just to push the rest of the fuel. Our derivation was for a simple rocket, but you can toss in all the sophisticated break-away fuel tanks and multi-stage boosters you like and it won't help much. In the end, you must still run head-first into that logarithm.