How To Sink A Pirate Ship

For a particle moving freely under the influence of gravity (no air friction, $g = 9.8$ m/s^2), we can assume that the motions in the x- and y-directions are independent. The most general equations for the x- and y-coordinates of a particle in this situation are:

$$y = y_o + v_y\, t - \tfrac{1}{2} g t^2 \quad \text{and} \quad x = x_o + v_x\, t$$

where x_o, y_o are the initial coordinates of the particle, and v_x, v_y are the initial (constant) velocities of the particle in the x- and y-directions.

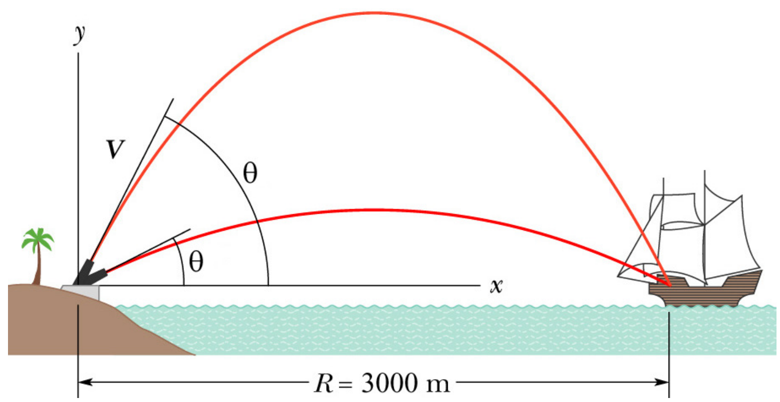
We can solve for t in the second equation to get $t = (x - x_o) / v_x$. We then substitute this expression for t into the first equation, and rearrange terms (after some simple but boring algebra) to get:

$$y = [y_o - (v_y / v_x)x_o - g\, x_o^2/(2v_x^2)] + [(v_y / v_x) + (g\, x_o / v_x^2)]\, x - (g / 2v_x^2)\, x^2 \qquad \text{(Equation 1)}$$

This solution has the basic form $y = a + bx + cx^2$, which is the general equation for a parabola. The formula is admittedly a bit complicated, but in most cases we can place the origin at the point where the particle is launched, i.e., we can set $x_o = y_o = 0$. The formula then simplifies greatly, to:

$$y = (v_y / v_x)\, x - (g / 2v_x^2)\, x^2 \qquad \text{(Equation 2)}$$

This formula is essentially the same as equation 3.27 on page 81 of the textbook (12th edition). Let's solve a problem to see how it works. The figure below shows a pirate ship. If I fire a cannonball with velocity "v" at a pirate ship which is a distance "R" away, at what angle θ do I have to aim my cannon to hit the ship?



We recognize that the correct solution has $y = 0$ because the ship is at the same height as the cannon. The solution must also have $x = R$. Substituting these into Equation 2:

$0 = (v_y / v_x)\, R - (g / 2v_x^2)\, R^2$, and a bit of algebra yields $2v_xv_y = gR$.

For converting to polar coordinates, we know that $v_x = v \cos\theta$ and $v_y = v \sin\theta$, because if the velocity "v" of the cannon ball is considered to be the hypotenuse of a right triangle then v_x and v_y must be the two sides. This yields $2v^2\sin\theta \cos\theta = gR$, and using the well-known trig identity $\sin(2\theta) = 2\sin\theta \cos\theta$ brings us to $\sin(2\theta) = gR/v^2$.

We have $g = 9.8$ m/s^2 and $R = 3000$ m. We will set $v = 270$ m/s, which is around 600 mi/hr, and that's about right for an 18th-century cannonball. Then $\sin(2\theta) = (9.8)(3000)/(270)^2 = 0.4033$, which gives us $2\theta = 23.8°$, or $\theta = 11.9°$.

But that is not the end of the story. If we remember from our basic trig that $\sin(\psi) = \sin(180 - \psi)$, then it must be that $\sin(23.8°) = \sin(156.2°)$. [Go ahead. Put the numbers into your calculator and see for yourself.] Thus, a *second* solution to the problem is $2\theta = 156.2°$, or $\theta = 78.1°$. There are two ways to hit the pirate ship. One is to aim the cannon low and fairly straight, the other way is to angle it nearly upward and shoot a high "lob" shot. The two trajectories are illustrated in the figure.